ONE-PARAMETER GENERALIZATIONS OF RAMANUJAN'S FORMULA FOR π

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Abstract. Several terminating generalizations of Ramanujan's formula for $\frac{1}{\pi}$ with complete WZ proofs are given.

One of Ramanujan's [2] infinite series representation for $\frac{1}{\pi}$ is the series

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{2}{\pi} \quad . \qquad (Ramanujan)$$

Zeilbeger [5] gave a short WZ proof of (*Ramanujan*) by first proving a one-parameter generalization, namely

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^2(-n)_k}{k!^2(\frac{3}{2}+n)_k}=\frac{\Gamma(\frac{3}{2}+n)}{\Gamma(\frac{3}{2})\Gamma(n+1)} \quad , \qquad (Zeilberger)$$

of (*Ramanujan*) for *nonnegative* integers n using WZ method, and then evaluating both sides of the identity at $n=-\frac{1}{2}$, thanks to Carlson's theorem (see below).

In this article, following Zeilberger's approach, we provide several more one-parameter generalizations of (*Ramanujan*) complete with their WZ proofs. These generalizations (identities) are of interest on their own right as they appear to be new at least for us. But first,

Notation: We denote a hypergeometric series

$$_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};z,p(k)\right)=\sum_{k=0}^{\infty}p(k)\frac{(a)_k(b)_k(c)_k}{k!(d)_k(e)_k}z^k \quad ,$$

by $F(a,b,c;d,e;z,p(k))$, where $(a)_j=\frac{\Gamma(a+j)}{\Gamma(a)}$ and p is a polynomial in k.

Observe that the above series can also be viewed as a $_4F_3$ hypergeometric series. The following well-known theorem due to Carlson is used to justify that if an identity holds for *positive* integers, then it also holds for rational arguments under suitable conditions.

Key words and phrases. Ramanujan's [2] Formula, Hypergeometric Series, Infinite Series Representations, $\frac{1}{\pi}$.

Theorem : (Carlson [2]) If $f(z)$ is analytic and is $0\left(e^{k|z|}\right)$, where $k<\pi$, for $Re(z)\geq 0$, and if $f(z)=0$ for $z=0,1,2,\ldots$, then $f(z)$ is identically zero.

Theorem 1 :

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(-n)_k^2(\frac{1}{2})_k}{k!\left(n+\frac{3}{2}\right)_k^2}=\left(\frac{1}{4}\right)^n\frac{\left(\frac{3}{2}\right)_n^2}{\left(\frac{5}{4}\right)_n\left(\frac{3}{4}\right)_n}\quad .$$

Proof :

Let $F(n,k)$ be the summand divided by the right hand side of the equality. Construct $G(n,k)=R(n,k)F(n,k)$, where $R(n,k)$ is the rational function (certificate)

$$R(n,k)=-\frac{(6n^2+10n+4+k-2k^2)k}{(n-k+1)^2(4k+1)}\quad ,$$

so that $F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k)$. Now sum both sides of this last equation with respect to k ($k=0$ to $k=\infty$), to see that the right hand side telescopes to zero from which it follows that $\sum F(n,k)=Constant$. Finally, plugging in $n=0$, we get $\sum_{k=0}^{\infty}F(n,k)=1$ completing the WZ proof of the theorem for *nonnegative* integers n. To deduce (*Ramanujan*), substitute $n=-\frac{1}{2}$ which is legitimate by Carlson's theorem. QED

In our notation, the statement of theorem 1 is equivalent to

$$F(-n,-n,\frac{1}{2};1,n+\frac{3}{2};-1,4k+1)=\left(\frac{1}{4}\right)^n\frac{\left(\frac{3}{2}\right)_n^2}{\left(\frac{5}{4}\right)_n\left(\frac{3}{4}\right)_n}\quad .$$

Below we provide more terminating generalizations which reduces to (*Ramanujan*) when evaluated at $n=-\frac{1}{2a}$, where a is the coefficient of n in $F(-an,b,c;d,e;z,p(k))$. In the remaining generalizations except theorem 2, the right hand side do not automatically simplify to $\frac{2}{\pi}$ which by itself gives some interesting relationship between different Gamma and trigonometric values. To wit, in theorem 6 below, when we evaluate the right hand side of the identity at $n=-\frac{1}{2}$, we get $\frac{\sqrt{5}}{\pi(\cos(\frac{\pi}{5})+\cos(\frac{2\pi}{5}))}$ which equals $\frac{2}{\pi}$ (To see $\cos(\frac{\pi}{5})+\cos(\frac{2\pi}{5})=\frac{1}{2}\sqrt{5}$, consider roots of $4x^2-2x-1=0$).

Theorem 2 :

$$F\left(-n,-2n-\frac{1}{2};\frac{1}{2},n+\frac{3}{2};2n+2,-1,4k+1\right)=\left(\frac{2^2}{3^3}\right)^n\frac{\left(\frac{3}{2}\right)_n^2}{\left(\frac{4}{3}\right)_n\left(\frac{2}{3}\right)_n}\quad .$$

Proof : Let

$R(n,k)=(184n^4+658n^3-44k^2n^2+22kn^2+868n^2+38kn-76k^2n+500n+106+4k^4+17k-4k^3-33k^2)\times$

$$\frac{2k}{(4k+1)(4n+5-2k)(4n+3-2k)(-k+n+1)(2n+2+k)}\quad ,$$

and proceed as in Theorem 1.

Theorem 3 :

$$F\left(-2n,-n+\frac{1}{4},\frac{1}{2};n+\frac{5}{4},2n+\frac{3}{2};-1,4k+1\right)=\left(\frac{2^2}{3^3}\right)^n\frac{\left(\frac{5}{4}\right)_n^2}{\left(\frac{13}{12}\right)_n\left(\frac{5}{12}\right)_n}\quad .$$

Proof : Let

$R(n,k) = (2944n^4+7584n^3+352kn^2-704k^2n^2+7096n^2+432kn-864k^2n+2846n+64k^4+142k-64k^3-268k^2+411)\times$

$$\frac{-k}{4\left(4k+1\right)\left(2n-k+2\right)\left(2n-k+1\right)\left(4n+3-4k\right)\left(4n+3+2k\right)}\quad ,$$

and proceed as in Theorem 1.

Theorem 4:

$$F\left(-n,-3n-1,\frac{1}{2};n+\frac{3}{2},3n+\frac{5}{2};-1,4k+1\right)=\left(\frac{3^3}{2^8}\right)^n\frac{\left(\frac{7}{6}\right)_n\left(\frac{5}{6}\right)_n\left(\frac{3}{2}\right)_n^2}{\left(\frac{5}{8}\right)_n\left(\frac{7}{8}\right)_n\left(\frac{9}{8}\right)_n\left(\frac{11}{8}\right)_n}\quad .$$

Proof : Let

$R(n,k) := (8470+58774n+963k+210k^4-8460k^2n+7138kn^2+4286kn-1872k^2+251126n^3+12k^5-215k^3+208908n^4-8k^6-10528k^2n^3+1452kn^4+5264kn^3-14214k^2n^2+91488n^5-448k^3n+167522n^2-2904k^2n^4-248k^3n^2+16488n^6+248k^4n^2+448k^4n)\times$

$$\frac{-k}{\left(4k+1\right)\left(3n+4-k\right)\left(3n+3-k\right)\left(3n+2-k\right)\left(-k+n+1\right)\left(6n+7+2k\right)\left(6n+5+2k\right)}$$

and proceed as in Theorem 1.

Theorem 5 :

$$F\left(-3n,-n+\frac{1}{3},\frac{1}{2};n+\frac{7}{6},3n+\frac{3}{2};-1,4k+1\right)=\left(\frac{3^3}{2^8}\right)^n\frac{\left(\frac{5}{6}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{7}{6}\right)_n^2}{\left(\frac{25}{24}\right)_n\left(\frac{7}{24}\right)_n\left(\frac{13}{24}\right)_n\left(\frac{19}{24}\right)_n}\quad .$$

Proof : Let

$R(n,k) = (33736+412476n+12183k+7146k^4-167112k^2n+230202kn^2+86418kn-22458k^2+5023782n^3+972k^5-7551k^3+6796548n^4-648k^6-539136k^2n^3+117612kn^4+269568kn^3-455382k^2n^2+4739472n^5-22896k^3n+2011788n^2-235224k^2n^4-20088k^3n^2+1335528n^6+20088k^4n^2+22896k^4n)\times$

$$\frac{-k/\left(27\left(4k+1\right)\right)}{\left(3n+2-3k\right)\left(3n+3-k\right)\left(3n+2-k\right)\left(3n-k+1\right)\left(6n+5+2k\right)\left(6n+3+2k\right)}$$

and proceed as in Theorem 1.

Theorem 6 :

$$F\left(-n,-4n-\frac{3}{2},\frac{1}{2};n+\frac{3}{2},4n+3;-1,4k+1\right)=\left(\frac{2^8}{5^5}\right)^n\frac{\left(\frac{5}{4}\right)_n\left(\frac{3}{4}\right)_n\left(\frac{3}{2}\right)_n^2}{\left(\frac{6}{5}\right)_n\left(\frac{7}{5}\right)_n\left(\frac{3}{5}\right)_n\left(\frac{4}{5}\right)_n}\quad .$$

Proof : Let

$R(n,k) := (1360680+12454594n+118956k+22445k^4-1561554k^2n+2189040kn^2+792996kn-232109k^2+110282888n^3+1528k^5+22139008n^7-23084k^3+152482100n^4+2937856n^8-2048k^6n+1656k^5n^2+3072k^5n+16k^8-1000k^6-6375798k^2n^3-1104k^6n^2+1133632kn^5+203392kn^6+2617070kn^4-406784k^2n^6+3201812kn^3-4339096k^2n^2+133488542n^5-97496k^3n+49304668n^2-2267264k^2n^5-5226636k^2n^4-32k^7-111304k^3n^3-155798k^3n^2+72279728n^6-30016k^3n^4+111304k^4n^3+155108k^4n^2+96216k^4n+30016k^4n^4))\times$

$$\frac{-2k/((4k+1)(8n+11-2k)(8n+9-2k))}{(8n+7-2k)(8n+5-2k)(-k+n+1)(4n+5+k)(4n+k+4)(4n+k+3)}$$

and proceed as in Theorem 1.

Theorem 7 :

$$F\left(-3n,-2n+\frac{1}{6},\frac{1}{2};2n+\frac{4}{3},3n+\frac{3}{2};-1,4k+1\right)=\left(\frac{2^2 3^3}{5^5}\right)^n\frac{\left(\frac{5}{6}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{7}{6}\right)_n^2}{\left(\frac{7}{15}\right)_n\left(\frac{13}{15}\right)_n\left(\frac{16}{15}\right)_n\left(\frac{4}{15}\right)_n}\quad .$$

Proof : Let

$R(n,k) = (4701560+76180392n+1024494k+708705k^4-22123845k^2n+56137239kn^2+11776977kn-1840161k^2+2005650450n^3+248832k^5+2986094808n^7-814086k^3+4671194832n^4+633425184n^8-520992k^6n+717336k^5n^2+781488k^5n+11664k^8-152280k^6-277628958k^2n^3-478224k^6n^2+144102888kn^5+43337592kn^6+196729884kn^4-86675184k^2n^6+141103782kn^3-108466425k^2n^2+6791227920n^5-5655312k^3n+524305530n^2-288205776k^2n^5-391357332k^2n^4-23328k^7-18314424k^3*n^3-15172434k^3n^2+6025575744n^6-8409744k^3n^4+18314424k^4n^3+14873544k^4n^2+5329692k^4n+8409744k^4n^4)\times$

$$\frac{-2k/(27(4k+1)(12n+11-6k)(12n+5-6k))}{(3n+3-k)(3n+2-k)(3n-k+1)(6n+5+2k)(6n+3+2k)(6n+4+3k)}$$

and proceed as in Theorem 1.

Theorem 8 : $F\left(-4n,-n+\frac{3}{8},\frac{1}{2};n+\frac{9}{8},4n+\frac{3}{2};-1,4k+1\right)=\left(\frac{2^8}{5^5}\right)^n\frac{\left(\frac{7}{8}\right)_n\left(\frac{3}{8}\right)_n\left(\frac{9}{8}\right)_n^2}{\left(\frac{33}{40}\right)_n\left(\frac{41}{40}\right)_n\left(\frac{9}{40}\right)_n\left(\frac{17}{40}\right)_n}$.

Proof : Let

$R(n,k) = (12815055+232274998n+4590732k-396544k^6-1249280k^6n-1130496k^6n^2+28224057344n^5+7320264608n^3+18289568000n^4+26350223360n^6+54254040kn+208273408kn^6+942632960kn^4+3008364544n^8+671602688kn^3+692224000kn^5+263857056kn^2-32768k^7-103023312k^2n-8360336k^2-416546816k^2n^6-1877581824k^2n^4-1326237696k^2n^3-1384448000k^2n^5-513366592k^2n^2+21002112k^4n+2969696k^4+$

$30736384k^4n^4+67870720k^4n^3+56542208k^4n^2-21782912k^3n-3231872k^3-30736384k^3n^4-67870720k^3n^3-57248768k^3n^2+16384k^8+1761550336n^2+13645250560n^7+1873920k^5n+623488k^5+1695744k^5n^2)\times$

$$\frac{-k/(128(4k+1)(8n+5-8k)(4n+4-k))}{(4n+3-k)(4n-k+2)(4n-k+1)(8n+7+2k)(8n+5+2k)(8n+3+2k)}$$

Theorem 9 :

$$F\left(-2n,-3n-\frac{1}{4},\frac{1}{2};2n+\frac{3}{2},3n+\frac{7}{4};-1,4k+1\right)=\left(\frac{2^2 3^3}{5^5}\right)^n \frac{\left(\frac{11}{12}\right)_n\left(\frac{7}{12}\right)_n\left(\frac{5}{4}\right)_n^2}{\left(\frac{11}{20}\right)_n\left(\frac{19}{20}\right)_n\left(\frac{23}{20}\right)_n\left(\frac{7}{20}\right)_n}\quad.$$

Proof : Let

$R(n,k)=(22623909+306149258n+3494086k+14411873792n^5+5772117536n^3+11505823872n^4+232853504kn^5+367020544kn^4+305077760kn^3+34504208kn+60874752kn^6+141134368kn^2+11083683840n^6+889749504n^8-6486428k^2-46570e008k^2n^5-731087872k^2n^4-602739712k^2n^3-65967264k^2n-121749504k^2n^6-275188800k^2n^2-1968960k^3-11812864k^3n^4-29663232k^3n^3-12059136k^3n-28235776k^3n^2+1779904k^4+29663232k^4n^3+11812864k^4n^4+11531776k^4n+27815936k^4n^2+448000k^5+1265664k^5n+1007616k^5n^2+1775873160n^2+4787625984n^7-279552k^6-843776k^6n-671744k^6n^2-32768k^7+16384k^8)\times$

$$\frac{-k/(4(4k+1)(12n+13-4k)(12n+9-4k)(12n+5-4k))}{(2n-k+2)(2n-k+1)(4n+3+2k)(12n+11+4k)(12n+7+4k)}$$

and proceed as in theorem 1.

A similar proof can be constructed for the following two identities using Zeilberger algorithm.

Theorem 10 :

$$F\left(-4n,-3n+\frac{1}{8},\frac{1}{2};3n+\frac{11}{8},4n+\frac{3}{2};-1,4k+1\right)=\left(\frac{2^8 3^3}{7^7}\right)^n \frac{\left(\frac{11}{24}\right)_n\left(\frac{3}{8}\right)_n\left(\frac{7}{8}\right)_n\left(\frac{19}{24}\right)_n\left(\frac{9}{8}\right)_n^2}{\left(\frac{11}{56}\right)_n\left(\frac{43}{56}\right)_n\left(\frac{19}{56}\right)_n\left(\frac{51}{56}\right)_n\left(\frac{27}{56}\right)_n\left(\frac{59}{56}\right)_n}\quad.$$

Theorem 11 :

$$F\left(-3n,-4n-\frac{1}{6},\frac{1}{2};3n+\frac{3}{2},4n+\frac{5}{3};-1,4k+1\right)=\left(\frac{2^8 3^3}{7^7}\right)^n \frac{\left(\frac{11}{12}\right)_n\left(\frac{5}{6}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{5}{12}\right)_n\left(\frac{7}{6}\right)_n^2}{\left(\frac{11}{21}\right)_n\left(\frac{23}{21}\right)_n\left(\frac{5}{21}\right)_n\left(\frac{17}{21}\right)_n\left(\frac{8}{21}\right)_n\left(\frac{20}{21}\right)_n}\quad.$$

Conclusion

In this article we considered one parameter generalizations of one of the many formulas of Ramanujan for π. It would be interesting to find if similar generalizations exist for other similar formulas for $\frac{1}{\pi}$. For example a notable one is the series

$$2\sqrt{2}\sum_{k=0}^{\infty}\left(\frac{1}{99}\right)^{4k+2}(1103+26390k)\frac{(\frac{1}{4})_k(\frac{1}{2})_k(\frac{3}{4})_k}{k!^3}=\frac{1}{\pi} \quad .$$

See http://mathworld.wolfram.com/PiFormulas.html for complete list of similar formulas.

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